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A segmented least-squares algorithm is presented which solves for biases, bias rates and higher-order terms of intersecting satellite altimetry tracks. Input data are geoid-height differences at the intersections. Step-by-step instructions for coding this algorithm are provided in an appendix.

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FOREWORD

This report presents an algorithm for extracting ephemeris errors from satellite radar altimetry data. The algorithm is currently used at the Naval Surface Weapons Center, Dahlgren Laboratory for reducing GEOS-3 altimetry data to geoid heights and vertical deflections. The report was reviewed by R. J. Anderle, Head, Astronautics Division, and by C. J. Cohen, Research Associate, Warfare Analysis Department.

Appreciation is extended to Alan C. Chappell for coding and implementing this algorithm, and to Ronald J. Koenecke and Ted Sahlin for check-out and assistance on an earlier version of the algorithm. Discussions with P.J. Fell which helped to illuminate some of the theoretical aspects of this problem are appreciated.

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I. INTRODUCTION

The accuracy of geoid heights and vertical deflections derived from satellite altimetry depends primarily on the accuracy of the satellite's ephemeris and on the measurement noise. The measurement noise is for most part a short-wavelength (< 1 km) random process; it is usually taken out by filtering or smoothing the along-track geoid heights.[1] The ephemeris errors[2] on the other hand, consist mainly of long-wavelength (> 1000 km) biases and bias rates. These can be solved for and eliminated if a sufficient number of track intersections is available in a given region. The data for this solution are the geoid-height discrepancies (differences of the two filtered geoid heights) at the intersections.

This report presents an approximate least-squares algorithm for such a solution. It is an iterative algorithm in the sense that first it solves for biases alone. Subtracting the effect of this solution from the original (0th-order) discrepancies leaves a set of 1st-order discrepancies. These are then used to solve for bias rates, leaving 2nd-order discrepancies, which are used to solve for 2nd-order effects, etc. This process can be terminated at the user's discretion.

It is easy to show that this iterative procedure does not give the optimum least-squares solution. For example, if the algorithm is terminated with the second-order solution, the result will not be as good as a least-squares solution with the full parameter set consisting of biases, bias rates and 2nd-order terms. If biases alone are desired, then this algorithm of course will give the correct least-squares solution. It is difficult to quantify by how much the iterative, segmented solution is degraded. Test cases show that for most applications this will be negligible. It has the great advantages, however, that it is relatively simple and requires much less computer time and storage. This comes from the fact that the matrices to be inverted in this algorithm have dimensions which depend only on the number of tracks, and not on the number of tracks times the number of bias parameters to be solved for, as would be the case with the full least-squares solution.

The general algorithm, which is derived in chapter 2, takes account of missing data points (where no intersections exist) by giving them zero weight. All other data are weighted equally. The problem is inherently singular with infinitely many solutions. The singularities are lifted by assigning a-priori variances for the bias parameters on each track. The track biases are assumed to be uncorrelated. In chapter 3 the solution is presented for biases only when there are no missing data points. This solution turns out to be very simple; no matrix inversions are needed.

In chapter 4, several test cases are presented which will be useful for check-out purposes. Finally, Appendix A gives a step-by-step procedure for coding the algorithm on a computer.

II. DERIVATION OF THE ALGORITHM

A. THE OBSERVATION EQUATIONS

Figure 1 shows a network of $n + m$ intersecting tracks which consist of n "rows," $1 \leqslant i \leqslant n$, and m "columns," $1 \leqslant j \leqslant m$. The index i will always denote a row, and j a column.

We thus have a total of $n \cdot m$ intersections. At each of these intersection points (i, j) we either have a data point Δ_{ij}, or the data point is missing in which case an asterisk is placed at that intersection. (In practice a missing data point means that there is no intersection.)

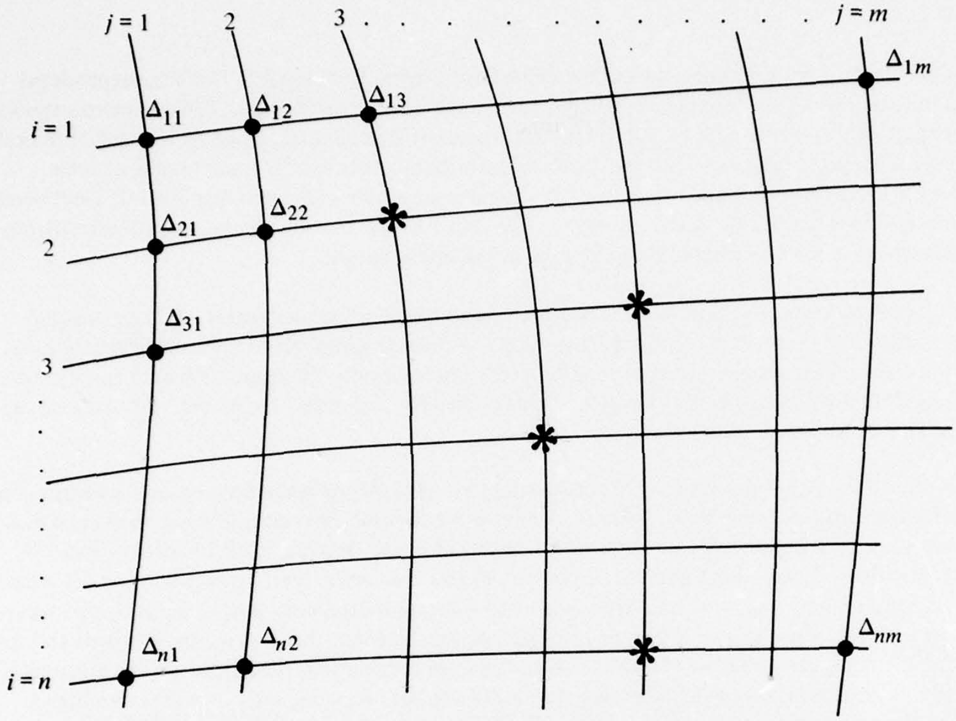

Figure 1. Intersection Data Δ_{ij}

The model for the biases on each track (row or column) is assumed to be a power series in τ – the time along the track, with $\tau = 0$ at the middle of the track:

$$b(\tau) = b + \dot{b}\tau + \ddot{b}\tau^2 + \dots b^{(k)}\tau^k \tag{1}$$

We shall first solve for the constant biases alone. Subtracting their contribution from the original data $\Delta_{ij}^{(0)}$ will give a new set of residuals $\Delta_{ij}^{(1)}$. This new set of data will then be used to solve for the bias rates $\dot{b}$, yielding 2nd-order residuals $\Delta_{ij}^{(2)}$, etc., until any desired order k.

The observation equations for a least-squares solution of any order k are given by

$$\Delta_{ij}^{(k)} = x_i U_{ji} - y_j V_{ij},$$
$$1 \leqslant i \leqslant n, \tag{2}$$
$$1 \leqslant j \leqslant m,$$

where the unknowns x_i, y_j, and the constants U_{ji}, V_{ij} are defined in Table 1.

The times τ in this table are:

$$\tau_{ji}^{(R)} = j^{\text{th}} \text{ time on } i^{\text{th}} \text{ row},$$

$$\tau_{ij}^{(c)} = i^{\text{th}} \text{ time on } j^{\text{th}} \text{ column}.$$

Table 1. The Unknowns x_i, y_j and U_{ji}, V_{ij} of Equation (2)

Solution Order	x_i	y_j	U_{ji}	V_{ij}
0 (bias)	b_i	b_j	1	1
1 (bias rate)	$\dot{b}_i$	$\dot{b}_j$	$\tau_{ji}^{(R)}$	$\tau_{ij}^{(c)}$
2	$\ddot{b}_i$	$\ddot{b}_j$	$[\tau_{ji}^{(R)}]^2$	$[\tau_{ij}^{(c)}]^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$b_i^{(k)}$	$b_j^{(k)}$	$[\tau_{ji}^{(R)}]^k$	$[\tau_{ij}^{(c)}]^k$

B. THE NORMAL EQUATIONS

The normal equations for the least-squares solution of Equation (2) with arbitrary weighting W are

$$A^T WA \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = A^T W \Delta, \tag{3}$$

where Δ is the $(n \cdot m) \times 1$ data vector formed from the residuals $\Delta_{ij}^{(k)}$. A is the $(n \cdot m) \times (n + m)$ matrix of partial derivatives

$$\begin{aligned} \frac{\partial \Delta_k}{\partial x_i} &= U_{ji}, \\ \frac{\partial \Delta_k}{\partial y_j} &= -V_{ij}, \\ 1 \leqslant k \leqslant n \cdot m, \quad & 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant m. \end{aligned} \tag{4}$$

It is written out explicitly in Table 2. The weighting matrix W will be used here only to accommodate missing data points, although it can be implemented in general as an arbitrary weighting matrix for all of the data. All existing data Δ_{ij} will be given unit weight. Accordingly, let W be the $(n \cdot m) \times (n \cdot m)$ diagonal matrix with

$$\begin{aligned} W_{kk} &= 1 \text{ for } k = (i, j) \text{ where } \Delta_{ij} \text{ exists}, \\ &= 0 \text{ for } k = (i, j) \text{ where } \Delta_{ij} \text{ is missing}. \end{aligned}$$

Table 2. The A Matrix

	·	$i = 1 \quad 2 \quad 3 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; n$	$j = 1 \quad 2 \quad 3 \; . \; . \; . \; . \; . \; m$
$i = 1$	$j = 1$	U_{11}	$-V_{11}$
	2	U_{21}	$-V_{12}$
	3	U_{31}	$-V_{13}$
	$\vdots$		
	m	U_{m1}	$-V_{1m}$
$i = 2$	1	$0 \; U_{12}$	$-V_{21}$
	2	$0 \; U_{22}$	$-V_{22}$
	3	$0 \; U_{32}$	$-V_{23}$
	$\vdots$	$\vdots$	
	m	$0 \; U_{m2}$	$-V_{2m}$
$i = 3$	1	$0 \; 0 \; U_{13}$	$-V_{31}$
	2	$0 \; 0 \; U_{23}$	$-V_{32}$
	3	$0 \; 0 \; U_{33}$	$-V_{33}$
	$\vdots$	$\vdots \; \vdots \; \vdots$	
	m	$0 \; 0 \; U_{m3}$	$-V_{3m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$i = n$	1	U_{1n}	$-V_{n1}$
	2	U_{2n}	$-V_{n2}$
	3	U_{3n}	$-V_{n3}$
	$\vdots$	$\vdots$	
	m	U_{mn}	$-V_{nm}$

It is easy to show that this leads to

$$WA = A, \quad A^T W = A^T \tag{5}$$

providing that we redefine U_{ji} and V_{ij} such that

$$U_{ji} = V_{ij} \equiv 0 \tag{6}$$

at all intersections (i, j) with missing data points, and leave them as given in Table 1 for points (i, j) where Δ_{ij} does exist. Equation (3) now becomes

$$A^T A \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = A^T \Delta. \tag{7}$$

C. THE SOLUTION

The matrix $A^T A$ in Equation (7) is singular. For the bias solution ($k = 0$ in Equation (2)), the reason for that singularity is obvious: we may add to all of the biases x_i and y_j in Equation (2) an arbitrary constant without changing the data vector Δ_{ij}. For the higher-order solutions similar singularities can be established. The problem, therefore, has infinitely many solutions. In order to obtain a unique solution we lift these singularities by imposing a-priori constraints $\sigma_{i,R}$ and $\sigma_{j,C}$ for the row and column biases, respectively. The solution of Equation (7) then is

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = B^{-1} A^T \Delta \tag{8}$$

with

$$B = A^T A + \Sigma, \tag{9}$$

where Σ is the diagonal matrix of the reciprocal a-priori variances with elements

$$\Sigma_{kl} = \frac{1}{\sigma_k^2} \delta_{kl}. \tag{10}$$

With the A matrix as given in Table 2, it is easy to obtain the B matrix as

$$B = \left[\begin{array}{c|c} \Lambda_{(n \times n)} & Q_{(n \times m)} \\ \hline Q^T_{(m \times n)} & M_{(m \times m)} \end{array}\right] \tag{11}$$

where Λ and M are diagonal matrices

$$\Lambda_{ik} = \left(\sum_{j=1}^{m} U_{ji}^2 + \frac{1}{\sigma_{i,R}^2} \right) \delta_{ik}, \tag{12a}$$

$$M_{jl} = \left(\sum_{i=1}^{n} V_{ij}^2 + \frac{1}{\sigma_{j,C}^2} \right) \delta_{jl}, \tag{12b}$$

and

$$Q_{ij} = - V_{ij} U_{ji}. \tag{12c}$$

The range of indices in Equations (12) are as shown by the dimensions of the four matrices in Equation (11).

The inverse of the B matrix is

$$B^{-1} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \tag{13}$$

with

$$W = (\Lambda - QM^{-1}Q^T)^{-1}, \tag{13a}$$

$$Z = (M - Q^T\Lambda^{-1}Q)^{-1}, \tag{13b}$$

$$X = -\Lambda^{-1}QZ, \tag{13c}$$

$$Y = -M^{-1}Q^TW. \tag{13d}$$

Note that $Y = X^T$, but we prefer to leave them in this form for symmetry reasons. The dimension of the largest matrix to be inverted (W or Z) is either $(n\times n)$ or $(m\times m)$, the number of rows or columns. These inverses will in general have to be done on the computer. In the case there are no missing data points, they can be done analytically for the 0th-order (biases) solution (see the next chapter).

We may write the solution, Equation (8), in a little more expanded form. The $(n\times 1)$ *Row Solutions* are

$$\mathbf{x} = W\mathbf{s}^{(R)} + Xs^{(C)}, \tag{14a}$$

and the $(m\times 1)$ *Column Solutions* are

$$\mathbf{y} = Y\mathbf{s}^{(R)} + Zs^{(C)}, \tag{14b}$$

where

$$\begin{pmatrix} \mathbf{s}^{(R)} \\ \mathbf{s}^{(C)} \end{pmatrix} = A^T\mathbf{\Delta}. \tag{15}$$

The vectors $\mathbf{s}^{(R)}$ and $\mathbf{s}^{(C)}$ are appropriately weighted sums of the data residuals on each row and column, viz:

$$s_i^{(R)} = \sum_{j=1}^{m} \Delta_{ij}U_{ji}, \qquad 1 \leqslant i \leqslant n; \tag{16a}$$

$$s_j^{(C)} = -\sum_{i=1}^{n} \Delta_{ij}V_{ij}, \qquad 1 \leqslant j \leqslant m, \tag{16b}$$

Equations (14) with (12), (13) and (16) constitute the complete solution in matrix form. In Appendix A the algorithm is given in component form, which is suitable for direct coding on a computer.

III. SOLUTION FOR BIASES WITH A FULL DATA SET

When there are no missing data points Δ_{ij} the solution for biases alone can be obtained in closed form, i.e., the B matrix, Equation (11), can be inverted in closed form. This is a big advantage over the general solution, so that in applications where only the constant biases are needed, it may be advisable to fill in the missing data points.

With a full data set, for the bias solution,

$$U_{ji} = V_{ij} = 1, \tag{17}$$

so that Equation (12c) becomes

$$Q_{ij} = -1, \tag{18}$$

which may be written as

$$Q = -\mathbf{u}\mathbf{v}^T, \tag{19}$$

where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}_{n\times 1}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{m\times 1} \tag{20}$$

are the $(n\times 1)$ and $(m\times 1)$ constant vectors whose components are all unity. We have the relations

$$\mathbf{u}^T\mathbf{u} = n, \qquad \mathbf{v}^T\mathbf{v} = m. \tag{21}$$

We specialize this derivation a little as compared with the general solution, by not allowing arbitrary a-priori variances. Rather, we require that all rows have the same a-priori variance σ_R^2 and all columns another, σ_C^2. The more general case, with different σ's for each track, can be handled in closed form also, but requires much more algebra. Equations (12a) and (12b) now become

$$\begin{aligned} &\Lambda = \lambda I_{n\times n}, \qquad M = \mu I_{m\times m}; \\ &\lambda = m + \frac{1}{\sigma_R^2}, \quad \mu = n + \frac{1}{\sigma_C^2}; \end{aligned} \tag{22}$$

so that Equations (13a) and (13b) can be written as

$$W = \frac{1}{\lambda}\left(I - \frac{m}{\lambda\mu}\,\mathbf{u}\mathbf{u}^T\right)^{-1}, \tag{23a}$$

$$Z = \frac{1}{\mu}\left(I - \frac{n}{\lambda\mu}\,\mathbf{v}\mathbf{v}^T\right)^{-1}. \tag{23b}$$

These matrices can be inverted by using the general identity for any two vectors **a** and **b**

$$(I \pm \mathbf{a}\mathbf{b}^T)^{-1} = I \mp \frac{\mathbf{a}\mathbf{b}^T}{1 \pm \mathbf{a}^T\mathbf{b}} \cdot \tag{24}$$

The result is

$$W = \frac{1}{\lambda}\left(I + \frac{m}{d}\,\mathbf{u}\mathbf{u}^T\right), \tag{25a}$$

$$Z = \frac{1}{\mu}\left(I + \frac{n}{d}\,\mathbf{v}\mathbf{v}^T\right), \tag{25b}$$

$$d = \lambda\mu - mn. \tag{25c}$$

Equations (13c) and (13d) yield

$$X = \frac{1}{d}\,\mathbf{u}\mathbf{v}^T; \qquad Y \approx \frac{1}{d}\,\mathbf{v}\mathbf{u}^T. \tag{26}$$

The solution, Equations (14), can now be readily evaluated to be

Row Biases:

$$b_i^{(R)} = \frac{1}{\lambda}\,s_i^{(R)} + \frac{m}{\lambda d}\,S_R + \frac{1}{d}\,S_C, \qquad 1 \leqslant i \leqslant n \tag{27a}$$

Column Biases:

$$b_j^{(C)} = \frac{1}{\mu}\,s_j^{(C)} + \frac{n}{\mu d}\,S_C + \frac{1}{d}\,S_R, \qquad 1 \leqslant j \leqslant m \tag{27b}$$

where Equations (16) give

$$s_i^{(R)} = \sum_{j=1}^{m} \Delta_{ij}, \quad s_j^{(C)} = -\sum_{i=1}^{n} \Delta_{ij}; \tag{28}$$

and where we have defined

$$S_R \equiv \sum_{i=1}^{n} s_i^{(R)}, \quad S_C \equiv \sum_{j=1}^{m} s_j^{(C)}. \tag{29}$$

Note that the full data set Δ_{ij} must satisfy the constraint

$$S \equiv S_R = -S_C. \tag{30}$$

With this, the solution, Equation (27), assumes a particularly simple form:

Row Biases:

$$b_i^{(R)} = \frac{1}{\lambda}\left(s_i^{(R)} - \frac{S}{\sigma_R^2 d}\right), \quad 1 \leqslant i \leqslant n; \tag{31a}$$

Column Biases:

$$b_j^{(C)} = \frac{1}{\mu}\left(s_j^{(C)} + \frac{S}{\sigma_C^2 d}\right), \quad 1 \leqslant j \leqslant m. \tag{31b}$$

IV. NUMERICAL EXAMPLES

We present here several numerical examples. These may be used as test cases for checking the algorithm out after it is programmed on a computer. The first four have only six intersections, so that they are simple enough for hand calculations. The solutions as well as some intermediate results are given. The last two examples consist of 50 rows and 60 columns.

A. EXAMPLE 1

$n = 3$ rows, $\quad m = 2$ columns.

A-priori statistics: $\quad \sigma_R = \sigma_C = 3.$

We construct data for this example by assuming that the five tracks have the following true biases:

$$\mathbf{b}_{\text{TRUE}}^{(R)} = \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix}; \quad \mathbf{b}_{\text{TRUE}}^{(C)} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}. \tag{32}$$

The data matrix $\Delta_{ij} \equiv b_i - b_j$ therefore is

$$\Delta^{(0)} = \begin{bmatrix} 1 & 6 \\ -7 & -2 \\ -4 & 1 \end{bmatrix}, \tag{33}$$

which has a standard deviation

$$\sigma_{\Delta^{(0)}} = 4.535. \tag{34}$$

Some intermediate results (see Equations (22) and (30)) are

$$\lambda = 2\,\frac{1}{9}\,, \quad \mu = 3\,\frac{1}{9}\,, \quad S = -5. \tag{35}$$

The solution, Equation (31), is:

$$\mathbf{b}^{(R)} = \begin{pmatrix} 3.779 \\ -3.800 \\ -.958 \end{pmatrix}, \quad \mathbf{b}^{(C)} = \begin{pmatrix} 2.900 \\ -1.922 \end{pmatrix}, \tag{36}$$

which does not compare well at all with the true biases given in Equation (32). The solution, of course, is completely insensitive to an overall bias, simply because the data $\Delta_{ij}^{(0)}$ are so also. We must compare, therefore, Equation (36) with the true biases minus their mean (which is $\bar{b} = 2$):

$$\mathbf{b}_{\mathrm{TRUE}}^{(R)} - \bar{\mathbf{b}} = \begin{pmatrix} 4 \\ -4 \\ -1 \end{pmatrix}; \quad \mathbf{b}_{\mathrm{TRUE}}^{(C)} - \bar{\mathbf{b}} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \tag{37}$$

Doing that, we find that the solution errors are -5.5, -5.0, -4.2, -3.3 and -3.9%, respectively for the five biases. This implies that our chosen a-priori σ was too small.

The 1st-order residuals, formed by subtracting the solution, Equation (36) from the data, Equation (33), are:

$$\Delta_{ij}^{(1)} = \Delta_{ij}^{(0)} - b_i + b_j,$$

$$\Delta^{(1)} = \begin{bmatrix} .121 & .299 \\ -.300 & -.122 \\ -.142 & .036 \end{bmatrix}, \tag{38}$$

which has the standard deviation

$$\sigma_{\Delta^{(1)}} = 0.214. \tag{39}$$

This, when compared with Equation (34), shows the relative efficacy of the solution to reduce the intersection discrepancies.

B. EXAMPLE 2

This is the same as Example 1, except that the a-priori variances are increased to

$$\sigma_R = \sigma_C = 10.$$

The solution is

$$\mathbf{b}^{(R)} = \begin{pmatrix} 3.979 \\ -3.981 \\ -.996 \end{pmatrix}, \quad \mathbf{b}^{(C)} = \begin{pmatrix} 2.991 \\ -1.993 \end{pmatrix}, \tag{40}$$

which now agrees much better with Equation (37).

The remaining 1st-order residuals are

$$\Delta^{(1)} = \begin{bmatrix} .012 & .028 \\ -.028 & -.012 \\ -.013 & .003 \end{bmatrix}, \tag{41}$$

with the standard deviation

$$\sigma_{\Delta^{(1)}} = .020. \tag{42}$$

C. EXAMPLE 3

Here we take the same data $\Delta^{(0)}$ as in the two previous examples, but delete the intersection of the second row with the first column. We keep the a-priori variances as in Example 2:

$$\sigma_R = \sigma_C = 10.$$

Because there is now a "missing data point," the general solution of section IIC must be used.

The data matrix is

$$\Delta^{(0)} = \begin{bmatrix} 1 & 6 \\ * & -2 \\ -4 & 1 \end{bmatrix}, \tag{43}$$

where the asterisk indicates the missing data point; it can be filled in with an arbitrary value. Equations (4) and (12) become

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad Q = -\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \tag{44a}$$

$$\Lambda = \begin{bmatrix} 2.01 & 0 & 0 \\ 0 & 1.01 & 0 \\ 0 & 0 & 2.01 \end{bmatrix}, \quad M = \begin{bmatrix} 2.01 & 0 \\ 0 & 3.01 \end{bmatrix}. \tag{44b}$$

Equations (13) yield

$$W^{-1} = \begin{bmatrix} 1.180262 & -.332226 & -.829738 \\ -.332226 & .677774 & -.332226 \\ -.829738 & -.332226 & 1.180262 \end{bmatrix}, \tag{45a}$$

$$Z^{-1} = \begin{bmatrix} 1.014975 & -.995025 \\ -.995025 & 1.024876 \end{bmatrix}. \tag{45b}$$

The inverses are

$$W = \begin{bmatrix} 20.387249 & 19.742660 & 19.889737 \\ 19.742660 & 20.830026 & 19.742660 \\ 19.889737 & 19.742660 & 20.387249 \end{bmatrix}, \tag{45c}$$

$$Z = \begin{bmatrix} 20.436714 & 19.841465 \\ 19.841465 & 20.239281 \end{bmatrix}. \tag{45d}$$

The rest of Equations (13) can now be evaluated as

$$X = \begin{bmatrix} 20.038895 & 19.940670 \\ 19.645015 & 20.038892 \\ 20.038895 & 19.940670 \end{bmatrix} \tag{46a}$$

$$Y = X^T. \tag{46b}$$

Equation (16) gives

$$\mathbf{s}^{(R)} = \begin{pmatrix} 7 \\ -2 \\ -3 \end{pmatrix}, \quad \mathbf{s}^{(C)} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \tag{47}$$

so that the solution, Equations (14), are

$$\mathbf{b}^{(R)} = \mathbf{x} = \begin{pmatrix} 3.970 \\ -3.949 \\ -1.006 \end{pmatrix}, \quad \mathbf{b}^{(C)} = \mathbf{y} = \begin{pmatrix} 2.968 \\ -1.987 \end{pmatrix}. \tag{48}$$

The 1st-order residuals are calculated as in Equation (38):

$$\Delta^{(1)} = \begin{bmatrix} -.002 & .043 \\ * & -.038 \\ -.026 & .019 \end{bmatrix}. \tag{49}$$

Their standard deviation is

$$\sigma_{\Delta^{(1)}} = .033, \tag{50}$$

which is but slightly larger than the value in Equation (42) that is obtained with the full data set.

It is interesting to see the effect on the solution of "filling in" the missing data point in Equation (43). One possible procedure is to replace the missing point by one half the sum of the two averages on the defective row and column. (For a larger data grid one would not average over the entire row and column, but only over points near the missing data point). Doing this we get, instead of Equation (43),

$$\Delta^{(0)} = \begin{bmatrix} 1 & 6 \\ -1.75 & -2 \\ -4 & 1 \end{bmatrix}. \tag{51}$$

Since now again we have a full data set, we may use the much simpler solution of chapter 3. Keeping the same a-priori variances $\sigma_R = \sigma_C = 10$, the solution is

$$\mathbf{b}^{(R)} = \begin{pmatrix} 3.458 \\ -1.890 \\ -1.517 \end{pmatrix}, \quad \mathbf{b}^{(C)} = \begin{pmatrix} 1.595 \\ -1.645 \end{pmatrix}, \tag{52}$$

with resulting 1st-order residuals

$$\Delta^{(1)} = \begin{bmatrix} -.863 & .897 \\ 1.736 & -1.754 \\ -.888 & .872 \end{bmatrix}, \tag{53}$$

whose standard deviation is

$$\sigma_{\Delta^{(1)}} = 1.356. \tag{54}$$

This is much worse than the corresponding results, Equations (49) and (50), where the missing data point was not filled in but was given zero weight in the solution. This is undoubtedly due to the very small data size. For larger data grids, with relatively fewer missing data points, the degradation of the solution due to filling in the missing points is not expected to be so severe.

D. EXAMPLE 4

In this example biases as well as bias rates are considered. To each of the biases of Example 1 we add bias rates as shown in Figure 2. Remember that the time origin $\tau = 0$ with respect to which the bias rate and higher-order effects propagate along each track must be placed at the middle of the grid. The data matrix computed from Figure 2 is

$$\Delta^{(0)} = \begin{bmatrix} 1 & 5 \\ -6.5 & -3 \\ -3.5 & 1.5 \end{bmatrix}, \tag{55}$$

which has the standard deviation

$$\sigma_{\Delta^{(0)}} = 4.164. \tag{56}$$

D1. Bias Solution

As in Example 2, we take for the a-priori bias standard deviations

$$\sigma_R = \sigma_C = 10.$$

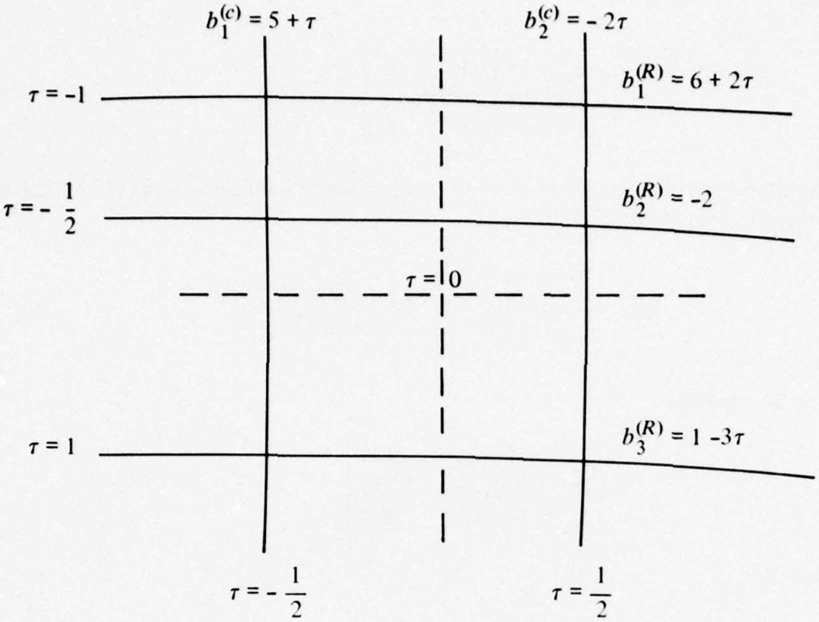

Figure 2. Biases and Bias Rates for Example 4

The solution, Equations (31), is

$$\mathbf{b}^{(R)} = \begin{pmatrix} 3.531 \\ -4.180 \\ -.449 \end{pmatrix}, \quad \mathbf{b}^{(C)} = \begin{pmatrix} 2.625 \\ -1.528 \end{pmatrix}, \tag{57}$$

which leaves the 1st-order residuals

$$\Delta^{(1)} = \begin{bmatrix} .094 & -.059 \\ .305 & -.348 \\ -.426 & .421 \end{bmatrix} \tag{58}$$

with a standard deviation

$$\sigma_{\Delta^{(1)}} = .342. \tag{59}$$

D2. Bias Rate Solution

The 1st-order residuals, Equation (58), are now used to solve for the bias rates. We assume a-priori values on bias rates of

$$\sigma_R^{(1)} = \sigma_C^{(1)} = 5.$$

The U and V matrices (see Table 1) are readily constructed from Figure 2.

$$U = \begin{bmatrix} -.5 & -.5 & -.5 \\ .5 & .5 & .5 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & -1 \\ -.5 & -.5 \\ 1 & 1 \end{bmatrix}, \tag{60}$$

whence Equation (12c) yields

$$Q = -\frac{1}{4}\begin{bmatrix} 2 & -2 \\ 1 & -1 \\ -2 & 2 \end{bmatrix}. \tag{61}$$

Some of the intermediate results, Equations (12) and (13) are:

$$\Lambda = .54\, I_{3\times 3}, \quad M = 2.29\, I_{2\times 2}; \tag{62}$$

$$W^{-1} = \begin{bmatrix} .321659 & -.109170 & .218341 \\ -.109170 & .485415 & .109170 \\ .218341 & .109170 & .321659 \end{bmatrix}; \tag{63a}$$

$$Z^{-1} = \begin{bmatrix} 1.248333 & 1.041667 \\ 1.041667 & 1.248333 \end{bmatrix}; \tag{63b}$$

$$W = \begin{bmatrix} 10.148762 & 4.148438 & -8.296910 \\ 4.148438 & 3.926063 & -4.148438 \\ -8.296910 & -4.148438 & 10.148762 \end{bmatrix}; \tag{63c}$$

$$Z = \begin{bmatrix} 2.637703 & -2.201022 \\ -2.201022 & 2.637703 \end{bmatrix}; \tag{63d}$$

$$X = \begin{bmatrix} 4.480301 & -4.480301 \\ 2.240150 & -2.240150 \\ -4.480301 & 4.480301 \end{bmatrix}; \tag{63e}$$

$$Y = X^T. \tag{63f}$$

Equations (16) yield the vectors

$$\mathbf{s}^{(R)} = \begin{pmatrix} -.0765 \\ -.3265 \\ .4235 \end{pmatrix}, \quad \mathbf{s}^{(C)} = \begin{pmatrix} .6725 \\ -.6540 \end{pmatrix}, \tag{64}$$

whereby the solutions, Equations (14), are obtained as follows:

Row bias rates:

$$\dot{\mathbf{b}}^{(R)} = \mathbf{x} = \begin{pmatrix} .299 \\ -.385 \\ .344 \end{pmatrix}; \tag{65a}$$

Column bias rates:

$$\dot{\mathbf{b}}^{(C)} = \mathbf{y} = \begin{pmatrix} .242 \\ -.234 \end{pmatrix}. \tag{65b}$$

These solutions compare very badly with the "true" bias rates as given in Figure 2:

$$\dot{\mathbf{b}}^{(R)}_{\text{TRUE}} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}, \quad \dot{\mathbf{b}}^{(C)}_{\text{TRUE}} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{66}$$

The reason is that the amount of data is very small. With only two intersections per row and three per column, one would not expect the data to have much strength for bias rate determination. The bias solution, Equation (57) took out too much of the residuals in this case.

Subtracting the effect of the bias rate solutions from Equation (58) leaves the 2nd-order residuals

$$\Delta^{(2)} = \begin{bmatrix} .002 & .025 \\ -.009 & -.038 \\ -.012 & .015 \end{bmatrix}, \tag{67}$$

which have a standard deviation

$$\sigma_{\Delta^{(2)}} = .022. \tag{68}$$

D3. The Correlation Matrix

It is instructive to look at the correlation matrix for this example. We define it as

$$R_{ii} = P_{ii} \equiv \sigma_i^2,$$
$$R_{ij} = \frac{P_{ij}}{\sigma_i \sigma_j}, \quad i \neq j, \tag{69}$$

where P is the usual covariance matrix

$$P = B^{-1}. \tag{70}$$

That is, the diagonal elements of R give the solution variances, while the off-diagonal elements are the correlation coefficients.

For the bias solution, Equation (57), the correlation matrix is

$$
\begin{array}{c}
\begin{array}{ccc|cc} & \text{Rows} & & \text{Columns} & \\ 1 & 2 & 3 & 1 & 2 \end{array} \\
R_b = \left[\begin{array}{ccc|cc}
20.358 & .976 & .976 & .984 & .984 \\
 & 20.358 & .976 & .984 & .984 \\
 & & 20.358 & .984 & .984 \\
\hline
 & & & 20.226 & .984 \\
 & & & & 20.266
\end{array}\right].
\end{array}
\tag{71}
$$

The diagonal elements show that the a-priori uncertainty of $\sigma^2 = 100$ has been reduced to approximately 20. This reduction was a little more effective for the column biases than for row biases, which is to be expected, since the columns have more intersection data than the rows. The correlation coefficients are all close to unity indicating that all biases may be in error by the same amount.

The correlation matrix for the bias rate solution, Equations (65), is more interesting:

$$
R_{\dot b} = \left[\begin{array}{ccc|cc}
10.149 & .657 & -.818 & .866 & -.866 \\
 & 3.926 & -.657 & .696 & -.696 \\
 & & 10.149 & -.866 & .866 \\
\hline
 & & & 2.638 & -.834 \\
 & & & & 2.638
\end{array}\right].
\tag{72}
$$

Now the a-priori variance $\sigma^2 = 25$ has been reduced to approximately 10 for the first and third rows, to 3.9 for the second row and to 2.6 for both columns. The correlation coefficients are not close to unity anymore. Thus, the element $R_{13} = -.818$ implies that, if the error in the bias rate solution for the first track is positive, the error for the third track will most probably be negative. Similarly, all the other elements can be qualitatively understood in terms of the geometry of Figure 2.

E. EXAMPLE 5

The previous examples had only five intersecting tracks. They are intended mainly to serve as check-out cases for a computer program. This and the next example are more realistic. They consist of a total of 3000 intersections with

$$n = 50 \text{ rows}, \qquad m = 60 \text{ columns}.$$

The data matrix was constructed by

$$\Delta_{ij}^{(0)} = b_i - b_j, \qquad \begin{array}{l} 1 \leqslant i \leqslant 50 \\ 1 \leqslant j \leqslant 60 \end{array} \tag{73}$$

where the 110 biases b_i and b_j were drawn from a Gaussian distribution with zero mean and standard deviation

$$\sigma_b = 5. \tag{74}$$

Two variations of this data grid were also constructed. The first had random noise δ_{ij} added to the $\Delta_{ij}^{(0)}$. This noise was generated from a uniform distribution $-.5 \leq \delta_{ij} \leq .5$, so that its standard deviation is

$$\sigma_\delta = \frac{1}{\sqrt{12}} = .289. \tag{75}$$

The second variation consisted in randomly deleting 26% of the data in Equation (73).

Instead of providing the a-priori variances σ_R^2 and σ_C^2 as input values, they are calculated in this as well as the next example by a prescription which is given in chapter 5.

The results of the solution for this example are given in Table 3 in terms of the standard deviations $\sigma_\Delta(k)$ for the kth-order residuals $\Delta^{(k)}$. The a-priori standard deviations σ_b, $\sigma_{\dot{b}}$ and $\sigma_{\ddot{b}}$, and the solution error standard deviation $\sigma_{\Delta b}$ are also given. With the full data set, and no noise, just the 0th-order solution (biases) reduces the residual standard deviation from 6.87 to .0013. The solution errors Δb have a standard deviation $\sigma_{\Delta b}$ = .00095. Additional, higher-order solutions reduce the residuals further by negligible amounts. This, of course, is expected, because the data consist of biases only. The third column of Table 3 shows the results for the noisy data. The solutions are able to reduce the residual variance only to the noise level, Equation (75). The last column is for the case when 26% of the data have been randomly deleted. Here the results show only a slight degradation as compared with the full data set.

Table 3. Standard Deviation $\sigma_{\Delta(k)}$ of kth-Order Residuals $\Delta^{(k)}$ for Example 5

k	Full Data, No Noise	Full Data With Noise	26% Data Deleted, No Noise
0	6.86827	6.87238	6.86084
1	.00133	.28229	.00182
2	.00133	.27562	.00182
3	.00133	.27314	.00181
Solution Error $\sigma_{\Delta b}$	$9.53 \cdot 10^{-4}$	.03769	.00131
A-Priori Standard Deviations σ_b	9.713	9.719	9.703
A-Priori Standard Deviations $\sigma_{\dot{b}}$	$1.22 \cdot 10^{-4}$	$2.58 \cdot 10^{-2}$	$1.66 \cdot 10^{-4}$
A-Priori Standard Deviations $\sigma_{\ddot{b}}$	$5.88 \cdot 10^{-6}$	$1.22 \cdot 10^{-3}$	$8.01 \cdot 10^{-6}$

F. EXAMPLE 6

This is the same as the previous example with the addition of bias rates. The bias rates were selected from a Gaussian distribution with zero mean and standard deviation = 0.1. The data grid is assumed to be uniform with unit time between neighboring intersections, so that a bias rate = 0.1 yields a total change of 5 units in the data on a column and 6 units on a row. The affect of the bias rates should therefore be approximately equal to that of the biases (see Equation 74).

Table 4 gives the results. This time, as expected, it takes the 1st-order solution (bias rates) to reduce the original data variance to an acceptable level. The full data set with noise is again reduced to the noise level, Equation (75), but the deleted data set now shows a much larger degradation than in the previous example. The solution errors are in general much larger than in the previous example without bias rates, but they are still at an acceptable level.

Table 4. Standard Deviation $\sigma_{\Delta^{(k)}}$ of kth-Order Residuals $\Delta^{(k)}$ for Example 6

k		Full Data, No Noise	Full Data With Noise	26% Data Deleted, No Noise
0		7.27954	7.28569	7.26447
1		2.37727	2.40096	2.34391
2		.00379	.27556	.22535
3		.00379	.27308	.15971
Solution Error	$\sigma_{\Delta b}$	.25521	.25738	.32427
	$\sigma_{\Delta \dot{b}}$	.01119	.01154	.01188
A-Priori Standard Deviations	σ_b	10.295	10.304	10.274
	$\sigma_{\dot{b}}$	.218	.220	.214
	$\sigma_{\ddot{b}}$	$1.67 \cdot 10^{-5}$	$1.22 \cdot 10^{-3}$	$9.94 \cdot 10^{-4}$

V. A-PRIORI VARIANCES

The a-priori variances σ_k^2 which are needed in Equation (10) have been assumed to be given input parameters in most of the previous examples. In practice this will be the case, because one should have some fairly accurate a-priori knowledge about the uncertainties of the orbit biases and bias rates. If, however, such knowledge is not available, the method given here can be used to estimate the a-priori variances from the data.

The original, 0th-order, residuals are assumed to be due to biases only, Equation (2):

$$\Delta_{ij}^{(0)} = b_i - b_j. \tag{76}$$

Assuming these biases to be uncorrelated, and their expectation to be zero

$$\mathsf{E}(b_i b_j) = \sigma_b^2 \delta_{ij}, \tag{77}$$

$$\mathsf{E}(b_i) = 0, \tag{78}$$

we get by taking the expectation of the square of Equation (76):

$$\sigma_b^2 = \frac{1}{2} \mathsf{E}\{[\Delta_{ij}^{(0)}]^2\} = \frac{1}{2} \sigma_{\Delta^{(0)}}^2. \tag{79}$$

The first order residuals are modeled by

$$\Delta_{ij}^{(1)} = \dot{b}_i \tau_{ji}^{(R)} - \dot{b}_j \tau_{ij}^{(C)}. \tag{80}$$

Again, assuming that the bias rates $\dot{b}_i$ also obey Equations (77) and (78), we have upon taking the expectation of the square of this equation:

$$\sigma_{\dot{b}}^2 = \frac{1}{2\,\overline{\tau^2}}\,\sigma_{\Delta^{(1)}}^2, \tag{81}$$

where $\overline{\tau^2}$ is some mean over rows and columns of the squares of the intersection times.

In the same manner we derive the a-priori variances for the higher-order solutions:

$$\sigma_k^2 = \frac{1}{2\,\overline{\tau^{2k}}}\,\sigma_{\Delta^{(k)}}^2. \tag{82}$$

It must be stressed that the above relations are recommended for use only when there is no independent a-priori knowledge available. In particular they do not allow different a-priori values for different rows and columns, which may be desirable in a practical situation.

The a-priori σ's which were used in Examples 5 and 6 are actually twice as large as the values given by these expressions. We found by solving these examples with different a-priori σ's that the solutions (biases, bias rates etc.) are not sensitive to the a-priori values. The residuals, however, (excepting the noisy data), are quite sensitive. Doubling the a-priori variances gave us a little better residuals. Another reason for taking slightly larger a-priori variances than the best available estimate is that our segmented method, because it neglects all correlations between the different-order parameters, will always produce an optimistic covariance matrix P, Equation (70). Increasing the a-priori variances will correct this error somewhat.

VI. CONCLUSION

A method has been presented for extracting biases, bias rates and higher-order terms from intersecting satellite altimetry tracks. This method consists of segmenting the least-squares solution such that at each stage only one set of terms is being solved for. It is clear that an error is committed in this approach because the correlations between terms of different order are not taken into account.

In order to estimate this error, Example 6, with 110 tracks having both biases and bias rates, was solved by a minimum-norm least-squares method.[3] In that solution the model was not segmented; i.e., the biases and bias rates were solved for simultaneously. That method guarantees the absolute minimum residual variance, while constraining the parameter vector to have the smallest length. For the perfect data case, column 2 in Table 4, that solution gave a residual $\sigma = 9.3 \cdot 10^{-12}$ (all of which is probably due to round-off) as compared to our .00379. For the noisy data, column 3, the result was .27554 compared to our .27556. Furthermore, our error variances $\sigma_{\Delta b}$ and $\sigma_{\Delta \dot{b}}$ are acceptably small. (In the particular minimum-norm solution used they came out to be much larger than ours). This shows that the segmented algorithm is able to reduce the residuals very close to the optimum level. We have also shown that we can approach this optimum level arbitrarily close by increasing the a-priori variances. However, in order to obtain realistic covariances, it is recommended to keep the a-priori values consistent with actual a-priori knowledge.

The advantages of this algorithm far outweigh the theoretical objections due to the segmentation of the problem. These advantages are (1) flexibility of the algorithm and (2) computer cost savings. Flexibility is obtained because one can increase the number of parameters by going to higher-order solutions without increasing the complexity of the problem and, more important, without increasing computer storage requirements. As an example of cost savings, the minimum-norm solution in Example 6 required 125 sec CPU time on the CDC 6700 computer, whereas the segmented algorithm took only 55 sec. This ratio will continue to favor the segmented algorithm with increasing number of higher-order parameters as the square of that number.

It is expected that at the end of the GEOS-3 mission there will be available from 5000 to 10000 satellite altimetry tracks. Clearly, even the segmented algorithm would not be able to handle a job of this magnitude. Rather than a global, world-wide solution, we therefore recommend to solve smaller patches at a time. Continuity of the biases and higher-order terms between patches would be assured by requiring some overlap of neighboring patches.

An example of the use of this algorithm with real geodetic data can be found in Reference 4.

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APPENDIX A

COMPUTER FORMULATION FOR THE LEAST-SQUARES ALGORITHM

1. INTRODUCTION

This formulation gives step-by-step instructions for coding the general solution (Equations (14), (12), (13) and (16)) in component form rather than in the matrix form which is given in the text. Refer to Figure 1 for notation.

2. INPUT DATA

A. n = number of rows
m = number of columns

$$\left.\begin{array}{l} \sigma_k^{(R)} = \text{a-priori for rows} \\ \\ \sigma_k^{(C)} = \text{a-priori for columns} \end{array}\right\} \begin{array}{l} k = 0: \text{ biases} \\ k = 1: \text{ bias rates} \\ k = 2: \text{ 2nd-order} \\ \vdots \\ \text{etc.} \end{array}$$

$$\left.\begin{array}{l} \Delta_{ij}^{(0)} = \text{geoid-height discrepancy} \\ \quad \text{at } (i,j)^{\text{th}} \text{ intersection } \bullet \\ \\ \quad = XXX \text{ (anything) at points } * \\ \quad \text{where no intersection exists} \end{array}\right\} \begin{array}{l} 1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m \end{array}$$

B. For bias rates and higher-order solutions, you will also need the following data:

$T_{ji}^{(R)}$ = j^{th} time on i^{th} row

$T_{ij}^{(C)}$ = i^{th} time on j^{th} column

$T_{i,I}^{(R)}$ = first intersection time on i^{th} row

$T_{i,F}^{(R)}$ = last intersection time on i^{th} row

$T_{j,I}^{(C)}$ = first intersection time on j^{th} column

$T_{j,F}^{(C)}$ = last intersection time on j^{th} column

Shift these times, so that the new time $\tau \equiv 0$ is at the middle of each track:

$$\left.\begin{aligned} \tau_{ji}^{(R)} &= T_{ji}^{(R)} - \frac{1}{2}\left(T_{i,I}^{(R)} + T_{i,F}^{(R)}\right) \\ \tau_{ij}^{(C)} &= T_{ij}^{(C)} - \frac{1}{2}\left(T_{j,I}^{(C)} + T_{j,F}^{(C)}\right) \end{aligned}\right\} \text{at all intersections } \bullet \text{ where } \Delta_{ij} \text{ exist}$$

$$\tau_{ij}^{(C)} = \tau_{ji}^{(R)} \equiv 0 \text{ at all points } * \text{ where no } \Delta_{ij} \text{ exist}$$

3. PRELIMINARY CALCULATIONS

The *Mean of Residuals:*

$$\bar{\Delta}_0 = \frac{1}{N} \sum_i \sum_j \Delta_{ij}^{(0)}$$

The *Residual Variance:*

$$\sigma_0^2 = \frac{1}{N-1} \sum_i \sum_j \left(\Delta_{ij}^{(0)} - \bar{\Delta}_o\right)^2$$

Note:

a. The ranges of i and j in these two sums are only over those points $\bullet$ where Δ_{ij} exist. N = total numbers of such points.

b. The same formulas will be used for the means $\bar{\Delta}_k$ and variances σ_k^2 for the higher-order solutions.

c. Print $\bar{\Delta}_0$ and σ_0.

4. BIAS SOLUTION

A. Construct the $(m \times n)$ matrix U and the $n \times m$ matrix V:

$$\left.\begin{aligned} U_{ji} &= 1 \quad \text{at intersections } \bullet \\ &= 0 \quad \text{at all missing points } * \\ V_{ij} &= 1 \quad \text{at intersections } \bullet \\ &= 0 \quad \text{at } * \end{aligned}\right\} \begin{aligned} 1 \leq i \leq n \\ 1 \leq j \leq m \end{aligned}$$

Go To MAIN ALGORITHM $\Rightarrow x_i, y_j$

The solutions are:

$b_i^{(R)} = x_i$ = bias for i^{th} row, $\quad 1 \leqslant i \leqslant n$

$b_j^{(C)} = y_j$ = bias for j^{th} column, $\quad 1 \leqslant j \leqslant m$

B. *Bias Removal*

$\Delta_{ij}^{(1)} = \Delta_{ij}^{(0)} - x_i + y_j$ at all intersections $\bullet$.

Compute and print $\overline{\Delta}_1$ and σ_1 (see step 3).

5. BIAS RATE SOLUTION

A. Construct the matrices U and V:

$$\left.\begin{aligned} U_{ji} &= \tau_{ji}^{(R)} \\ V_{ij} &= \tau_{ij}^{(C)} \end{aligned}\right\} \begin{array}{l} 1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m \end{array}$$

Go To MAIN ALGORITHM $\Rightarrow x_i, y_j$

The solutions are:

$\dot{b}_i = x_i$ = bias rate for i^{th} row; $\quad 1 \leqslant i \leqslant n$

$\dot{b}_j = y_j$ = bias rate for j^{th} column; $\quad 1 \leqslant j \leqslant m$

B. *Bias Rate Removal*

$\Delta_{ij}^{(2)} = \Delta_{ij}^{(1)} - x_i U_{ji} + y_j V_{ij}$ at all intersections $\bullet$

Compute and print $\overline{\Delta}_2$ and σ_2.

6. HIGHER-ORDER SOLUTIONS, $k \geqslant 2$

A. *For k^{th}-order* ($k = 0$, biases, and $k = 1$, bias rates, were done in steps 4 and 5).

Construct the matrices U and V:

$$\left.\begin{aligned} U_{ji} &= (\tau_{ji}^{(R)})^k \\ V_{ij} &= (\tau_{ij}^{(C)})^k \end{aligned}\right\} \begin{array}{l} 1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m \end{array}$$

Go To MAIN ALGORITHM $\Rightarrow x_i, y_j$

$x_i = k^{\text{th}}$-order solution for i^{th} row; $\quad 1 \leqslant i \leqslant n$

$y_j = k^{\text{th}}$-order solution for j^{th} column; $\quad 1 \leqslant j \leqslant m$

B. *Remove k^{th}-order effect*

$\Delta_{ij}^{(k+1)} = \Delta_{ij}^{(k)} - x_i U_{ji} + y_j V_{ij}$ at all intersections.

Compute and print $\bar{\Delta}_{k+1}$ and σ_{k+1}

7. THE MAIN ALGORITHM (for k^{th} iteration)

$$\lambda_i = \sum_{j=1}^{m} U_{ji}^2 + \frac{1}{[\sigma_k^{(R)}]^2} \qquad 1 \leqslant i \leqslant n$$

$$\mu_j = \sum_{i=1}^{n} V_{ij}^2 + \frac{1}{[\sigma_k^{(C)}]^2} \qquad 1 \leqslant j \leqslant m$$

$$Q_{ij} = - V_{ij} U_{ji} \qquad \begin{matrix} 1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m \end{matrix}$$

$$(W^{-1})_{ik} = \lambda_i \delta_{ik} - \sum_{j=1}^{m} \frac{1}{\mu_j} Q_{ij} Q_{kj}; \qquad 1 \leqslant \begin{pmatrix} i \\ k \end{pmatrix} \leqslant n$$

$$(Z^{-1})_{jl} = \mu_j \delta_{jl} - \sum_{i=1}^{n} \frac{1}{\lambda_i} Q_{ij} Q_{il}; \qquad 1 \leqslant \begin{pmatrix} j \\ l \end{pmatrix} \leqslant m$$

INVERT the above symmetric matrices W^{-1} and Z^{-1}

$\Rightarrow$ the $(n \times n)$ matrix W

and the $(m \times m)$ matrix Z

The k^{th}-order solution is:

$$x_i = \sum_{k=1}^{n} \sum_{j=1}^{m} \left(W_{ik} U_{jk} + \frac{1}{\lambda_i} \sum_{l=1}^{m} Q_{il} Z_{lj} V_{kj} \right) \Delta_{kj}; \qquad 1 \leqslant i \leqslant n$$

$$y_j = - \sum_{l=1}^{m} \sum_{i=1}^{n} \left(Z_{jl} V_{il} + \frac{1}{\mu_j} \sum_{k=1}^{n} Q_{kj} W_{ki} U_{li} \right) \Delta_{il}; \qquad 1 \leqslant j \leqslant m$$

8. THE CORRELATION MATRIX

Calculate and print the following matrices:

$$\left(\begin{array}{c|c} R_{ik} & X_{ij} \\ \hline & C_{jl} \end{array}\right) \qquad \begin{array}{ll} 1 \leqslant i \leqslant n, & i \leqslant k \leqslant n \\ 1 \leqslant j \leqslant m, & j \leqslant l \leqslant m \end{array}$$

Row Correlations: $R_{ii} = W_{ii}$

$$R_{ij} = \frac{W_{ij}}{\sqrt{R_{ii} R_{jj}}}, \quad i \neq j$$

Column Correlations: $C_{jj} = Z_{jj}$

$$C_{jl} = Z_{jl} / \sqrt{C_{jj} C_{ll}}$$

Row-Column Correlations: $X_{ij} = - \frac{1}{\lambda_i} \left(\sum_{l=1}^{m} Q_{il} Z_{lj} \right) \Big/ \sqrt{R_{ii} C_{jj}}$

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CK-10	(20)
CK-12	(25)
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DX-40	